

# On the number of walks in a triangular domain

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## Abstract

We consider walks on a triangular domain that is a subset of the triangular lattice. We then specialise this by dividing the lattice into two directed sublattices with different weights. Our central result is an explicit formula for the generating function of walks starting at a fixed point in this domain and ending anywhere within the domain. Intriguingly, the specialisation of this formula to walks starting in a fixed corner of the triangle shows that these are equinumerous to two-coloured Motzkin paths, and two-coloured three-candidate Ballot paths, in a strip of finite height.

## 1 Introduction

Recently there has been significant development of the Kernel method [2, 3, 13], a technique in enumerative combinatorics. This method can be used to solve linear combinatorial functional equations in so-called catalytic variables.

While the Kernel method has been reasonably well understood if there is only one catalytic variable involved, once there are two or more catalytic variables the situation is far from clear. There are indications that the structure of the solution, such as whether the generating function is algebraic or perhaps not even differentiable finite, depends on the group of symmetries of the kernel of the functional equation [5, 11]. Only very recently has there been some progress using a many-variables Kernel method in a special case [4].

Based on the Kernel method, it is possible to derive generating functions for counting problems in previously inaccessible situations. As an example, the exact solution of a

lattice model of partially directed walks in a wedge has only been possible using an iterative version of the Kernel method [10], leading to a generating function that is not differentiable finite, as its singularities accumulate at limit points. This example also shows that as a by-product of enumerative combinatorics, deep combinatorial insight into connections between seemingly unrelated systems can be uncovered, leading to spin-off research in bijective combinatorics [12, 14].

In this paper we present a solution to an enumerative lattice path problem that is expressed in terms of a functional equation with three catalytic variables. We are able to solve this functional equation by virtue of the high symmetry of the kernel.

The resulting generating function solution allows us to prove a result linking walks on a triangular domain to Motzkin paths. Finding a bijective proof of this result poses an intriguing open problem.

## 2 Statement of Results

Consider walks  $(\omega_0, \omega_1, \dots, \omega_n)$  on  $\mathbb{Z}^{d+1}$  with steps  $\omega_i - \omega_{i-1}$  in a step-set  $\Omega$  such that with each step exactly one coordinate increases by one and exactly one coordinate decreases by one. More precisely,  $\Omega$  is the set of steps with coordinates  $(e_1, e_2, \dots, e_{d+1})$  such that for all ordered pairs  $(i, j)$  with  $1 \leq i, j \leq d+1$  and  $i \neq j$ ,  $e_i = 1$ ,  $e_j = -1$  and  $e_k = 0$  for all  $1 \leq k \leq d+1$  and  $k \notin \{i, j\}$ .

The step-set  $\Omega$  ensures that walks lie in a  $d$ -dimensional hyperplane  $\{(x_1, \dots, x_{d+1}) \in \mathbb{Z}^{d+1} \mid x_1, \dots, x_{d+1} = L\}$  determined by the starting point  $\omega_0 = (u_1, \dots, u_{d+1})$  of the walk, where  $L = \sum_{i=1}^{d+1} u_i$ . In this paper, walks on domains given by finite subsets of these hyperplanes are studied by restricting the walks to the non-negative orthant  $(\mathbb{N}_0)^{d+1}$ . Fixing the dimension  $d$ , this class of walks is referred to as the  $d$ -dimensional case. On  $\mathbb{Z}^2$  the domains are lines of length  $L$ , on  $\mathbb{Z}^3$  they are triangles of side-length  $L$ , and on  $\mathbb{Z}^4$  they are tetrahedra of side-length  $L$ .

Given a fixed starting point  $\omega_0$ , denote the number of  $n$ -step walks starting at  $\omega_0$  and ending at  $\omega_n = (i_1, \dots, i_{d+1})$  by  $C_n(i_1, \dots, i_{d+1})$  and consider the generating function

$$G(x_1, \dots, x_{d+1}; t) = \sum_{n=0}^{\infty} t^n \sum_{\omega_n \in (\mathbb{N}_0)^{d+1}} C_n(\omega_n) \prod_{j=0}^{j=d+1} x_j^{i_j}, \quad (1)$$

where  $t$  is the generating variable conjugate to the length of the walk. Due to the choice of the step-set  $\Omega$ ,  $G(x_1, \dots, x_{d+1}; t)$  is homogeneous of degree  $L = \sum_{i=1}^{d+1} u_i$  in  $x_1, \dots, x_{d+1}$ , i.e.

$$G(\gamma x_1, \dots, \gamma x_{d+1}; t) = \gamma^L G(x_1, \dots, x_{d+1}; t). \quad (2)$$

The 1-dimensional case has been studied before [6, pages 7-8], and has obvious connections to the generating function of Chebyshev polynomials. It is easy to solve, and gives the following result.

**Proposition 1.** *The generating function  $G(x, y; t)$ , which counts  $n$ -step walks starting at fixed  $\omega_0 = (u, v)$ , is given by*

$$G(x, y; t) = \frac{1}{1 - \frac{\frac{x}{y} + \frac{y}{x}}{p + \frac{1}{p}}} \left( x^u y^v - \frac{x^{u+v+1} p^{v+1} (1 - p^{2u+2})}{y(1 - p^{2u+2v+4})} - \frac{y^{u+v+1} p^{u+1} (1 - p^{2v+2})}{x(1 - p^{2u+2v+4})} \right), \quad (3)$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t} \quad (4)$$

is the generating function of Dyck paths.

This simplifies considerably when specifying  $x = y = 1$ .

**Corollary 2.** *The generating function  $G(1, 1; t)$ , which counts  $n$ -step walks starting at fixed  $\omega_0 = (u, v)$  with no restrictions on the endpoint, is given by*

$$G(1, 1; t) = \frac{(1 + p^2)(1 - p^{u+1})(1 - p^{v+1})}{(1 - p)^2(1 + p^{u+v+2})}, \quad (5)$$

where

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t}. \quad (6)$$

In this paper, the main result concerns the 2-dimensional case. Consider a weighted generalisation, where  $\Omega$  is partitioned into

$$\begin{aligned} \Omega' &= \{(1, -1, 0), (-1, 0, 1), (0, 1, -1)\} \quad \text{and} \\ \Omega'' &= \{(-1, 1, 0), (1, 0, -1), (0, -1, 1)\}, \end{aligned} \quad (7)$$

with steps in  $\Omega'$  and  $\Omega''$  given the weights  $\alpha$  and  $\beta$ , respectively. The main result of this paper is as follows.

**Theorem 3.** *The generating function  $G(t) \equiv G(1, 1, 1; t)$ , which counts  $n$ -step walks starting at fixed  $\omega_0 = (u, v, w)$  with no restrictions on the endpoint, is given by*

$$G(t) = \frac{(1 - p^3)(1 - p^{u+1})(1 - p^{v+1})(1 - p^{w+1})}{(1 - p)^3(1 - p^{u+v+w+3})}, \quad (8)$$

with

$$p = (\alpha + \beta)M((\alpha + \beta)t) \quad (9)$$

where

$$M(t) = \frac{1 - t - \sqrt{(1+t)(1-3t)}}{2t^2} \quad (10)$$

is the generating function of Motzkin paths.

For walks starting in a corner of a triangle of side-length  $L$  one finds the following intriguing equinumeracy results.

**Corollary 4.** (a) Walks starting in a corner of a triangle of side-length  $L = 2H + 1$  with arbitrary endpoint are in bijection with two-coloured Motzkin paths in a strip of height  $H$ .

(b) Walks starting at a corner of a triangle of side-length  $L = 2H$  with arbitrary endpoint are in bijection with two-coloured Motzkin paths in a strip of height  $H$ , such that horizontal steps at height  $H$  are forbidden.

**Corollary 5.** (a) Walks starting in a corner of a triangle of side-length  $L = 2H + 1$  with arbitrary endpoint, which only take steps on  $\Omega'$ , are in bijection with Motzkin paths in a strip of height  $H$ .

(b) Walks starting at a corner of a triangle of side-length  $L = 2H$  with arbitrary endpoint, which only take steps on  $\Omega'$ , are in bijection with Motzkin paths in a strip of height  $H$ , such that horizontal steps at height  $H$  are forbidden.

For walks starting in the centre of a triangle of side-length  $L = 3u$ , there is a further result.

**Proposition 6.** The generating function  $g(t) \equiv G(1, 1, 0; t)$ , which counts walks starting at  $\omega_0 = (u, u, u)$  and ending at a fixed side of the triangle, is given by

$$g(t) = p^u \frac{(1 - p^3)(1 - p^{u+1})}{(1 - p)(1 - p^{3u+3})}, \quad (11)$$

with  $p$  as in Theorem 3.

Define an  $n$ -step three-candidate Ballot path to be a walk  $(\omega_0, \omega_1, \dots, \omega_n)$  on  $\mathbb{N}_0^2$  starting at the origin with steps  $\omega_i - \omega_{i-1}$  taken from the step-set

$$\Delta = \{(1, 1), (1, -1), (1, 0)\}, \quad (12)$$

such that after  $r$  steps the number of  $(1, 1)$  steps is greater than or equal to the number of  $(1, -1)$  steps, which is greater than or equal to the number of  $(1, 0)$  steps, for all  $0 \leq r \leq n$ . These can also be thought of as a coding for Yamanouchi words with three letters [1, page 6]. Define further an  $n$ -step three-candidate Ballot path with excess  $L$  to be an  $n$ -step Ballot path such that after  $r$  steps the difference between the number of  $(1, 1)$  steps and  $(1, 0)$  steps is at most  $L$ , for all  $0 \leq r \leq n$ .

**Proposition 7.** Walks starting in a corner of a triangle of side-length  $L$  with arbitrary endpoint, restricted to the sublattice  $\Omega'$ , are in bijection with three-candidate Ballot paths with excess  $L$ .

### 3 Proofs

An  $n$ -step walk is uniquely constructed by appending a step from the step-set  $\Omega$  to an  $(n-1)$ -step walk, provided  $n > 0$ . This leads to the following functional equation for the generating function  $G(x_1, x_2, \dots, x_{n+1}; t)$ .

$$G(x_1, x_2, \dots, x_{d+1}; t) = \prod_{1 \leq i \leq d+1} x_i^{v_i} + G(x_1, x_2, \dots, x_{d+1}; t) t \sum_{i \neq j} \frac{x_i}{x_j} - \sum_{1 \leq i \leq d+1} \left( G(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{d+1}; t) \sum_{j \neq i} \frac{x_j}{x_i} \right) \quad (13)$$

Here, the monomial  $\prod_{1 \leq i \leq d+1} x_i^{v_i}$  corresponds to a zero-step walk starting (and ending) at  $\omega_0 = (v_1, v_2, \dots, v_{d+1})$ . The term  $G(x_1, x_2, \dots, x_{d+1}; t) t \sum_{i \neq j} \frac{x_i}{x_j}$  corresponds to appending any of the steps in  $\Omega$  irrespective of whether the resulting walk steps violates the boundary condition and leaves the domain. This overcounting is adjusted by the remaining term. For example,  $G(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{d+1}; t)$  corresponds to walks with end-points

$(v_1, \dots, v_{i-1}, 0, v_{i+1}, \dots, v_{d+1}; t)$ , and therefore  $G(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{d+1}; t) \sum_{j \neq i} \frac{x_j}{x_i}$  corresponds to precisely those walks stepping across the boundary.

As this is a functional equation for the generating function  $G(x_1, x_2, \dots, x_{d+1}; t)$  in the variables  $x_1, \dots, x_{d+1}$  only, the  $t$ -dependence is dropped by writing  $G(x_1, x_2, \dots, x_{d+1}; t) \equiv G(x_1, x_2, \dots, x_{d+1})$ . The functional equation (13) is further rewritten as

$$G(x_1, x_2, \dots, x_{d+1}) \left[ 1 - t \sum_{i \neq j} \frac{x_i}{x_j} \right] = \prod_{1 \leq i \leq d+1} x_i^{v_i} - \sum_{1 \leq i \leq d+1} \left( G(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{d+1}) \sum_{j \neq i} \frac{x_j}{x_i} \right) \quad (14)$$

Furthermore, the “Kernel”  $K(x, y, z; t) \equiv K(x, y, z)$  of the functional equation,

$$K(x_1, \dots, x_{n+1}) = 1 - t \sum_{i \neq j} \frac{x_i}{x_j}, \quad (15)$$

is introduced. Symmetry properties of this Kernel are central to the arguments. Note that the Kernel is homogeneous of degree zero, *i.e.* it is invariant under rescaling of all the variables. This trivial symmetry will be implicitly assumed in the considerations below.

Specialising to the 1-dimensional case, (14) gives the equation

$$G(x, y) \left[ 1 - t \left( \frac{x}{y} + \frac{y}{x} \right) \right] = x^u y^v - G(0, y) t \left( \frac{y}{x} \right) - G(x, 0) t \left( \frac{\beta x}{y} \right). \quad (16)$$

Furthermore, the Kernel (15) reduces to

$$K(x, y) = 1 - t \left( \frac{x}{y} + \frac{y}{x} \right) . \quad (17)$$

Now introduce  $G(S)$ , the group of transformations which leaves the kernel of the functional equation invariant for the step set  $S$ . This is in line with the notation introduced by Fayolle *et al.* [8]. For the 1-dimensional case, the step-set is

$$S_1 = \left\{ \frac{x}{y}, \frac{y}{x} \right\} ,$$

where here and henceforth, steps are identified with their associated combinatorial weights.

**Lemma 8.** *The Kernel  $K(x, y)$  is invariant under action of the group of transformations*

$$G(S_1) = \langle (y, x) \rangle \cong C_2 .$$

In particular, one arrives at the following result.

**Lemma 9.** *The Kernel  $K(x, y)$  is invariant under the following 1-parameter substitutions.*

$$K(p, 1) = K(1, p) = 1 - t \left( p + \frac{1}{p} \right) .$$

Substituting these two choices into the functional equation (16) and fixing the dependence between  $p$  and  $t$  such that

$$1 - t(p + 1/p) = 0 \quad (18)$$

implies

$$tpG(p, 0) + \frac{t}{p}G(0, 1) = p^u \quad (19a)$$

$$\frac{t}{p}G(1, 0) + tpG(0, p) = p^v . \quad (19b)$$

Using homogeneity of the generating function, replace

$$G(p, 0) = p^{u+v}G(1, 0) , \quad G(0, p) = p^{u+v}G(0, 1) , \quad (20)$$

and solve the two equations 19a and 19b in the two variables  $G(1, 0)$  and  $G(0, 1)$  to find that

$$G(1, 0) = \frac{p^{v+1}(p^{2u+2} - 1)}{t(p^{2u+2v+4} - 1)} , \quad G(0, 1) = \frac{p^{u+1}(p^{2v+2} - 1)}{t(p^{2u+2v+4} - 1)} . \quad (21)$$

Applying the homogeneity argument of (20) to (21), one determines  $G(x, 0)$  and  $G(0, y)$ , and substituting these into (16) and eliminating  $t$  via (18) gives the result stated in Proposition 1

$$G(x, y; t) = \frac{1}{1 - \frac{\frac{x}{y} + \frac{y}{x}}{p + \frac{1}{p}}} \left( x^u y^v - \frac{x^{u+v+1} p^{v+1} (1 - p^{2u+2})}{y(1 - p^{2u+2v+4})} - \frac{y^{u+v+1} p^{u+1} (1 - p^{2v+2})}{x(1 - p^{2u+2v+4})} \right), \quad (22)$$

and solving (18) for  $p$  gives the generating function for Dyck paths

$$p = \frac{1 - \sqrt{1 - 4t^2}}{2t} \quad (23)$$

as required. Substituting  $x = y = 1$  into (22) then gives Corollary 2

$$G(1, 1; t) = \frac{(1 + p^2)(1 - p^{u+1})(1 - p^{v+1})}{(1 - p)^2(1 + p^{u+v+2})}. \quad (24)$$

Specialising to the 2-dimensional case and including the weights  $\alpha$  and  $\beta$  corresponding to the directed sublattices (equation 7), (14) gives the equation

$$\begin{aligned} G(x, y, z) \left[ 1 - t \left( \frac{\beta x}{y} + \frac{\alpha y}{x} + \frac{\alpha x}{z} + \frac{\beta z}{x} + \frac{\beta y}{z} + \frac{\alpha z}{y} \right) \right] &= x^u y^v z^w \\ - G(0, y, z) t \left( \frac{\alpha y}{x} + \frac{\beta z}{x} \right) - G(x, 0, z) t \left( \frac{\beta x}{y} + \frac{\alpha z}{y} \right) - G(x, y, 0) t \left( \frac{\alpha x}{z} + \frac{\beta y}{z} \right) &. \end{aligned} \quad (25)$$

Then the Kernel (15) becomes

$$K(x, y, z) = 1 - t \left( \frac{\beta x}{y} + \frac{\alpha y}{x} + \frac{\alpha x}{z} + \frac{\beta z}{x} + \frac{\beta y}{z} + \frac{\alpha z}{y} \right). \quad (26)$$

In this case the step-set is

$$S_2 = \left\{ \frac{\beta x}{y} + \frac{\alpha y}{x} + \frac{\alpha x}{z} + \frac{\beta z}{x} + \frac{\beta y}{z} + \frac{\alpha z}{y} \right\},$$

and  $G(S)$  is generated by a rotation and an inversion.

**Lemma 10.** *The Kernel  $K(x, y)$  is invariant under action of the group of transformations*

$$G(S_2) = \left\langle (y, z, x), \left( \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \right) \right\rangle \cong C_3 \times C_2.$$

Moreover, there is a one-variable sub-set which has useful consequences.

**Lemma 11.** *The Kernel  $K(x, y, z)$  is invariant under the following 1-parameter substitutions.*

$$\begin{aligned} K(1, 1, p) &= K(1, p, 1) = K(p, 1, 1) = K(1, p, p) = K(p, 1, p) = K(p, p, 1) \\ &= 1 - t(\alpha + \beta) \left( p + 1 + \frac{1}{p} \right). \end{aligned} \quad (27)$$

Substituting these six choices into the functional equation (25) and fixing the dependence between  $p$  and  $t$  such that

$$1 - t(\alpha + \beta)(p + 1 + 1/p) = 0 \quad (28)$$

implies

$$\frac{(\alpha + \beta)t}{p}G(0, 1, 1) + t(\alpha + \beta p)G(p, 0, 1) + t(\alpha p + \beta)G(p, 1, 0) = p^u \quad (29a)$$

$$\frac{(\alpha + \beta)t}{p}G(1, 0, 1) + t(\alpha p + \beta)G(0, p, 1) + t(\alpha + \beta p)G(1, p, 0) = p^v \quad (29b)$$

$$\frac{(\alpha + \beta)t}{p}G(1, 1, 0) + t(\alpha + \beta p)G(0, 1, p) + t(\alpha p + \beta)G(1, 0, p) = p^w \quad (29c)$$

$$(\alpha + \beta)tpG(0, p, p) + t\left(\alpha + \frac{\beta}{p}\right)G(1, 0, p) + t\left(\frac{\alpha}{p} + \beta\right)G(1, p, 0) = p^vp^w \quad (29d)$$

$$(\alpha + \beta)tpG(p, 0, p) + t\left(\frac{\alpha}{p} + \beta\right)G(0, 1, p) + t\left(\alpha + \frac{\beta}{p}\right)G(p, 1, 0) = p^up^w \quad (29e)$$

$$(\alpha + \beta)tpG(p, p, 0) + t\left(\alpha + \frac{\beta}{p}\right)G(0, p, 1) + t\left(\frac{\alpha}{p} + \beta\right)G(p, 0, 1) = p^up^v \quad (29f)$$

Using homogeneity of the generating function, replace

$$G(p, p, 0) = p^L G(1, 1, 0), \quad G(p, 0, p) = p^L G(1, 0, 1), \quad G(0, p, p) = p^L G(0, 1, 1), \quad (30)$$

and from the linear combination  $[(29a) + (29b) + (29c)] - p[(29d) + (29e) + (29f)]$  it is easily found that

$$(\alpha + \beta)t[G(0, 1, 1) + G(1, 0, 1) + G(1, 1, 0)] = \frac{p^{u+1} + p^{v+1} + p^{w+1} - p^{2+L}(p^{-u} + p^{-v} + p^{-w})}{1 - p^{3+L}}. \quad (31)$$

Substituting  $(x, y, z) = (1, 1, 1)$  into (25) shows that  $G(1, 1, 1)$  can be computed explicitly, as

$$(1 - 3(\alpha + \beta)t)G(1, 1, 1) = 1 - (\alpha + \beta)t[G(0, 1, 1) + G(1, 0, 1) + G(1, 1, 0)]. \quad (32)$$

Substituting (31) into (32) and eliminating  $t$  via (28) gives the desired final result

$$G(1, 1, 1) = \frac{(1 - p^3)(1 - p^{u+1})(1 - p^{v+1})(1 - p^{w+1})}{(1 - p)^3(1 - p^{3+L})}. \quad (33)$$

Finally, note that substituting  $p(t) = (\alpha + \beta)tM((\alpha + \beta)t)$  followed by  $(\alpha + \beta)t = s$  into (28) implies

$$M(s) = 1 + sM(s) + s^2M(s)^2, \quad (34)$$

whence  $M(s)$  is the Motzkin path generating function. This completes the proof of Theorem 3.



Letting  $(u, v, w) = (L, 0, 0)$  in (8) implies that the generating function for walks starting in a corner is given by

$$G(1, 1, 1) = \frac{(1 - p^3)(1 - p^{1+L})}{(1 - p)(1 - p^{3+L})}. \quad (35)$$

This formula is intimately related to the convergents of the continued fraction expansion of the Motzkin path generating function. One can show by mathematical induction that in this case  $G(1, 1, 1)$  can be written as a continued fraction. More precisely, for  $L = 2H$  even, there is a continued fraction of length  $H$ ,

$$\underbrace{\cfrac{1}{1 - (\alpha + \beta)t - \cfrac{(\alpha + \beta)^2 t^2}{1 - (\alpha + \beta)t - \cfrac{(\alpha + \beta)^2 t^2}{\ddots - \cfrac{(\alpha + \beta)^2 t^2}{1 - (\alpha + \beta)t - (\alpha + \beta)^2 t^2}}}}_{\text{length } H} = \frac{(1 - p^3)(1 - p^{1+2H})}{(1 - p)(1 - p^{3+2H})}, \quad (36)$$

and for  $L = 2H + 1$  odd, there is a continued fraction of length  $H + 1$ ,

$$\underbrace{\cfrac{1}{1 - (\alpha + \beta)t - \cfrac{(\alpha + \beta)^2 t^2}{1 - (\alpha + \beta)t - \cfrac{(\alpha + \beta)^2 t^2}{\ddots - \cfrac{(\alpha + \beta)^2 t^2}{1 - (\alpha + \beta)t}}}}_{\text{length } H + 1} = \frac{(1 - p^3)(1 - p^{2+2H})}{(1 - p)(1 - p^{4+2H})}. \quad (37)$$

It is easy to show that equations (36) and (37) hold for the base case  $H = 0$ ,

$$1 = \frac{(1 - p^3)(1 - p^{1+0})}{(1 - p^{1+0})(1 - p^{3+0})} \quad (38a)$$

$$\frac{1}{1 - (\alpha + \beta)t} = \frac{(1 - p^3)(1 - p^{2+0})}{(1 - p)(1 - p^{4+0})}, \quad (38b)$$

and the inductive step follows from showing that

$$\frac{(1 - p^3)(1 - p^{1+(L+2)})}{(1 - p)(1 - p^{3+(L+2)})} = \frac{1}{1 - 2t - 4t^2 \frac{(1 - p^3)(1 - p^{1+L})}{(1 - p)(1 - p^{3+L})}}. \quad (39)$$

From the combinatorial theory of continued fractions the combinatorial interpretation in terms of Motzkin paths follows easily, as given in a paper by Flajolet [9, pages 6-11].

Substituting  $\alpha = 1$ ,  $\beta = 1$  gives coefficients 2 and 4, of  $t$  and  $t^2$  respectively, following from the fact that the relevant Motzkin paths are two-coloured. This immediately implies Corollary 4. Substituting in  $\alpha = 1$ ,  $\beta = 0$  gives the interpretation in terms of normal Motzkin paths, and Corollary 5 follows.

Attempting to solve the 2-dimensional case in full generality proved beyond the reach of the techniques used in this paper, as the system of equations (29) is underdetermined, linking nine quantities with six equations. The only case in which one can extract further information from it is one of high symmetry, namely when the starting point is chosen to be in the centre of the triangle, i.e.  $\omega_0 = (u, u, u)$ , in which the triangle has size  $L = 3u$ . The equations (29) then reduce to two equations in two unknowns,

$$\frac{(\alpha + \beta)t}{p}G(1, 1, 0) + (\alpha + \beta)t(1 + p)G(p, 1, 0) = p^u \quad (40a)$$

$$(\alpha + \beta)tp^{1+3u}G(1, 1, 0) + (\alpha + \beta)t\left(1 + \frac{1}{p}\right)G(p, 1, 0) = p^{2u}. \quad (40b)$$

This can readily be solved, and

$$(\alpha + \beta)tG(1, 1, 0) = \frac{p^{1+u}(1 - p^{1+u})}{1 - p^{3+3u}}. \quad (41)$$

Eliminating  $t$  by using (28) proves Proposition 6.

It now remains to prove our final result. Without loss of generality, starting in the corner marked by coordinates  $(L, 0, 0)$ , the steps  $(-1, 0, 1)$ ,  $(0, 1, -1)$ , and  $(1, -1, 0)$  can be mapped to  $(1, 1)$ ,  $(1, -1)$ , and  $(1, 0)$ , respectively. This maps steps in  $\omega'$  to steps in three-candidate Ballot paths and the restrictions imposed by the boundaries of the triangle clearly transfer to the restrictions on a Ballot path with excess  $L$ . This proves Proposition 7.

## 4 Conclusion and Open Problems

In this paper we have set up a general problem which we have completely solved in dimension 1. We have also solved it in dimension 2 in the case where endpoints are not weighted, along with a high-symmetry case. Unfortunately, our argument does provide enough information to solve the general case for dimension 2 case or that for higher dimensions and we must leave these open.

All of the generating functions we prove in this paper are rational. This is a direct result of the fact that the adjacency matrix for our system is of finite dimension. In particular, in the 2-dimensional case, a triangle of side-length  $L$  contains  $\binom{L+2}{2}$  vertices, or states, and therefore we would expect the degree of the numerator and denominator of the generating function to grow quadratically in  $L$ . However, due to some cancellation they grow linearly in  $L$ . For the general case in dimension 2, of walks with arbitrary fixed start and end points, we have some numerical evidence that the degrees of the numerator

and denominator grow quadratically in  $L$ , and it may be this extra complexity that has prevented us from solving this case with our method.

We have also proven intriguing equinumeracy results in Corollaries 4 and 5. Taking only steps on the directed sublattice  $\Omega'$  halves the out-degree of every vertex in the domain, and so it is clear that Corollary 4 implies Corollary 5. Eu [7] gives a bijective proof of Corollary 5 for the case of infinite side-length via standard Young tableaux (which are a coding of Yamanouchi words), Yeats [15] gives a bijective proof of Corollary 4 for the case of infinite side-length using intermediate markings, and the authors of this paper have bijective proofs of Corollary 4 for side-length  $L = 1, 2$  and 3. We note that Proposition 7 provides a possible alternative route to a bijective proof, via three-candidate Ballot paths. However, we have not been able to find a proof for general finite side-length, and therefore leave this as an open problem.

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